MATH 4030 Differential Geometry
Problem Set 5
due 17/11/2017 (Fri) at 5PM

Problems

(to be handed in)

1. Let $\alpha(t) : (-\epsilon, \epsilon) \to S$ be a curve on a surface $S \subset \mathbb{R}^3$. Suppose $X(t), Y(t)$ are two tangential vector fields defined along the curve α. Prove that

$$\frac{d}{dt}\langle X(t), Y(t)\rangle = \langle \nabla_{\alpha'(t)} X(t), Y(t)\rangle + \langle X(t), \nabla_{\alpha'(t)} Y(t)\rangle.$$

   Using this result, prove that the angle between two parallel vector fields X, Y along a curve is always constant.

2. Suppose $X(u,v) : U \to S \subset \mathbb{R}^3$ is an *orthogonal* parametrization of a surface S such that the first fundamental form is diagonal:

$$(g_{ij}) = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}.$$

   Show that the Gauss curvature is given by

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right].$$

   If, in addition, that X is an *isothermal* parametrization, i.e. $E = G = \lambda(u,v)$, then show that

$$K = -\frac{1}{2\lambda}\Delta(\log\lambda)$$

   where $\Delta := \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the standard Euclidean Laplace operator.

3. Prove that the surfaces parametrized by $(u,v) \in (0,+\infty) \times (0, 2\pi)$,

$$X(u,v) = (u\cos v, u\sin v, \log u)$$

$$\tilde{X}(u,v) = (u\cos v, u\sin v, v)$$

   have the same Gauss curvature at the points $X(u,v)$ and $\tilde{X}(u,v)$. However, show that the map $\tilde{X} \circ X^{-1}$ is not an isometry.

4. Compute all the Christoffel symbols of a surface of revolution parametrized by

$$X(u,v) = (f(v)\cos u, f(v)\sin u, g(v)), \quad (u,v) \in (0,2\pi) \times \mathbb{R}$$

   where $f, g : \mathbb{R} \to \mathbb{R}$ are smooth functions with $f > 0$ everywhere.

5. Explain why the saddle surface $\{z = x^2 - y^2\}$ is not locally isometric to any round sphere or cylinder.

6. Does there exist a parametrization $X(u,v) : U \to \mathbb{R}^3$ of a surface S such that the first and second fundamental forms are given by:

   (a) $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $(h_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;

   (b) $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}$ and $(h_{ij}) = \begin{pmatrix} \cos^2 u & 0 \\ 0 & 1 \end{pmatrix}$?

   Explain your answer.

Suggested Exercises

(no need to hand in)

1. Compute all the Christoffel symbols for the subset of the plane parametrized in

   (a) rectangular coordinates: $X(x,y) = (x,y,0)$, where $(x,y) \in \mathbb{R}^2$,

   (b) polar coordinates: $\tilde{X}(r,\theta) = (r\cos\theta, r\sin\theta, 0)$, where $(r,\theta) \in (0,+\infty) \times (0,2\pi)$.

2. Show that no neighborhood of a point on the unit sphere $\mathbb{S}^2$ is isometric to a subset of the plane.

3. Let $X(u^1,u^2) : U \to S \subset \mathbb{R}^3$ be a parametrization of a surface. Let ∂_1, ∂_2 be the coordinate vector field $\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}$ respectively and N be its associated unit normal. The first and second fundamental forms are (g_{ij}) and (A_{ij}) and Γ_{ij}^k denotes the Christoffel symbols. Let (g^{ij}) be the inverse matrix of (g_{ij}). The goal of this exercise is to show that the equation

$$\partial_\ell \partial_i N = \partial_i \partial_\ell N \tag{1}$$

   simply yields the Codazzi equation.

   (a) Use the Gauss and Weingarten equations to show that

$$\partial_\ell \partial_i N = [\partial_\ell (g^{kj} A_{ij}) + g^{pj}\Gamma_{\ell p}^k A_{ij}]\partial_k - (g^{pj} A_{\ell p} A_{ij})N.$$

   Hence, the normal components of (1) are automatically equal.

   (b) Prove that $\partial_\ell g^{kq} = -g^{qi} g^{kp} \partial_\ell g_{pi}$.

   (c) Use (b) and the formula of Γ_{ij}^k in terms of g_{ij} to show that

$$\partial_\ell g^{kq} + g^{pq}\Gamma_{\ell p}^k = -g^{kj}\Gamma_{j\ell}^q.$$

   (d) Use (a) and (c) to show that the tangential component of (1) gives the Codazzi equation.